

Lexicodes over Rings

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Abstract

In this paper, we consider the construction of linear lexicodes over \mathbb{Z}_4 and $\mathbb{F}_2 + u\mathbb{F}_2$ by using a B -ordering over these rings and a selection criteria. It is shown that this construction produces many optimal codes over rings and also good binary codes. Some of these codes meet the Gilbert bound. We also obtain optimal self-dual codes such as the octacode.

1 Introduction

Surprisingly, many good binary linear codes can be constructed using the following greedy algorithm with minimum distance as the selection criterion.

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Starting with the all zero vector, all binary vectors of length n are considered in lexicographic order, and when the distance of a vector to all other vectors in the code is at least δ , the vector is added to the code. Levenstein [18] proved that the resulting code (called lexicode), is linear. Conway and Sloane [9] proved that the lexicode is linear over fields of order 2^{2^l} , $l \in \mathbb{N}$. Moreover, they proved linearity when using a more general selection criterion called a turning-set. Brualdi and Pless [8] discussed another generalization of binary lexicode. They introduced the concept of a B -ordering, which is used in the greedy algorithm instead of the standard basis. Their starting point is a list of binary vectors of length n , ordered lexicographically with respect to a basis obtained by adding recursively all previous words to the next basis word. They also proved that the resulting lexicode is linear. Unfortunately, for fields other than \mathbb{F}_2 , the lexicode constructed using a B -ordering are not always linear. To solve this problem, Bonn [7] introduced another concept called forcing linearity. In this case, a list of all vectors over \mathbb{F}_q of length n is searched. This list need not be ordered in a specific way. If a vector \mathbf{a} satisfying $d(\mathbf{a}, \mathbf{y}) \geq \delta$ is found, then \mathbf{a} is added to the lexicode as well as all its multiples without checking the minimum distance condition. Surprisingly, the minimum distance condition is satisfied for all added words [7, Proposition 1]. Thus the resulting code, which is forced to be linear over all finite fields, has a basis composed of the selected vectors \mathbf{a} and has minimum distance greater than or equal to the designed distance δ .

Recently, van Zanten and Nengah Suparta [21, 22] generalized the work

of Bonn to a more general selection property over an arbitrary finite field \mathbb{F}_q . They considered a B -ordering on \mathbb{F}_q^n . The basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is ordered with respect to a lexicographically ordered list in a recursive way. By using a multiplicative selection property P , they proved that the resulting lexicode $C(B, P)$ is linear and such that each vector $\mathbf{x} \in C(B, P)$ satisfies the property P .

In this paper, we consider the construction of lexicode by using a B -ordering over the rings \mathbb{Z}_4 and $\mathbb{F}_2 + u\mathbb{F}_2$. Algorithms are given over both rings to find lexicode using multiplicative properties. The resulting codes are linear. As a special case, greedy algorithms are given to find self-orthogonal codes. In particular, the Octacode O_8 is obtained as a lexicode over \mathbb{Z}_4 using the minimum distance criteria. In this case we also prove that the corresponding binary image meets the Gilbert bound. We give tables of lexicode constructed over these rings by using several selective criteria. We compare the codes obtained over \mathbb{Z}_4 with the optimal codes in [2] and [12].

2 Preliminaries

We denote by \mathbb{Z}_4 the commutative ring with elements $\{0, 1, 2, 3\}$ and addition and multiplication modulo 4. The ring $\mathbb{F}_2 + u\mathbb{F}_2$, with $u^2 = 0$ is the finite commutative ring of characteristic 2 with elements $\{0, 1, u, \bar{u} = 1 + u\}$. Multiplication coincides with that of \mathbb{Z}_4 , while addition coincides with that of $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$, where w and w^2 are replaced by u and \bar{u} ,

respectively.

Let R be a commutative ring. Hence for an integer $n > 0$, R^n is an R -module. A non empty subset of R^n is said to be a linear code over R of length n if it is a submodule of R^n .

For $\mathbf{x} \in \mathbb{Z}_4^n$, denote the number of components of \mathbf{x} equal to a by $n_a(\mathbf{x})$. Hence the Hamming weight of \mathbf{x} is $wt_H(\mathbf{x}) = n_1(\mathbf{x}) + n_2(\mathbf{x}) + n_3(\mathbf{x})$. The Lee weight of \mathbf{x} is $wt_L(\mathbf{x}) = n_1(\mathbf{x}) + 2n_2(\mathbf{x}) + n_3(\mathbf{x})$, and the Euclidean weight of \mathbf{x} is $wt_E(\mathbf{x}) = n_1(\mathbf{x}) + 4n_2(\mathbf{x}) + n_3(\mathbf{x})$. For $\mathbf{x} \in (\mathbb{F}_2 + u\mathbb{F}_2)^n$, denote the number of components of \mathbf{x} equal to a by $n_a(\mathbf{x})$. Hence the Hamming weight of \mathbf{x} is $wt_H(\mathbf{x}) = n_1(\mathbf{x}) + n_u(\mathbf{x}) + n_{\overline{u}}(\mathbf{x})$. The Lee weight of \mathbf{x} is $wt_L(\mathbf{x}) = n_1(\mathbf{x}) + 2n_u(\mathbf{x}) + n_{\overline{u}}(\mathbf{x})$, and the Euclidean weight of \mathbf{x} is $wt_E(\mathbf{x}) = n_1(\mathbf{x}) + 4n_u(\mathbf{x}) + n_{\overline{u}}(\mathbf{x})$. The Hamming, Lee and Euclidean distances $d_H(\mathbf{x}, \mathbf{y})$, $d_L(\mathbf{x}, \mathbf{y})$, $d_E(\mathbf{x}, \mathbf{y})$ between two vectors \mathbf{x} and \mathbf{y} are $wt_H(\mathbf{x}-\mathbf{y})$, $wt_L(\mathbf{x}-\mathbf{y})$ and $wt_E(\mathbf{x}-\mathbf{y})$, respectively. The minimum Hamming, Lee and Euclidean weights, d_H , d_L and d_E of C are the smallest Hamming, Lee and Euclidean weights among all nonzero codewords of C . Any linear code over \mathbb{Z}_4 or $\mathbb{F}_2 + u\mathbb{F}_2$ has a generator matrix of the following form

$$G = \begin{pmatrix} I_{k_1} & A & B_1 + \gamma B_2 \\ O & \gamma I_{k_2} & \gamma C \end{pmatrix}$$

where $\gamma = 2$ for codes over \mathbb{Z}_4 and $\gamma = u$ for codes over $\mathbb{F}_2 + u\mathbb{F}_2$. The matrices A, B_1, B_2 and C have entries from \mathbb{F}_2 , and O is a $k_2 \times k_1$ zero matrix. The code C is said to be of type $4^{k_1}2^{k_2}$.

3 Construction of Lexicodes over \mathbb{Z}_4

The set \mathbb{Z}_4^n is a linear code over \mathbb{Z}_4 with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. With respect to this basis we recursively define a lexicographically ordered list $V_i = \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{4^i}$ as follows

$$V_0 := 0,$$

$$V_i := V_{i-1}, \quad \mathbf{b}_i + V_{i-1}, 2\mathbf{b}_i + V_{i-1}, 3\mathbf{b}_i + V_{i-1}, 1 \leq i \leq n.$$

In this way $|V_i| = 4^i$, and \mathbb{Z}_4^n is given by V_n . Assume now that we have a property P which can test if a vector $\mathbf{c} \in \mathbb{Z}_4^n$ is selected or not. Recall that a selection property P on V can be seen as a boolean valued function $P : V \longrightarrow \{\text{True}, \text{False}\}$ that depends on one variable. P is called a multiplicative property if $P[\mathbf{x}]$ is true implies $P[3\mathbf{x}]$ is true. Furthermore, assume that this property is multiplicative. The following greedy algorithm provides lexicodes over \mathbb{Z}_4^n .

Algorithm A

1. $C_0 := 0; i := 1;$
2. select the first vector $\mathbf{a}_i \in V_i \setminus V_{i-1}$ such that $P[\mathbf{a}_i + \mathbf{c}]$ and $P[2\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1};$
3. if such an \mathbf{a}_i exists, then $C_i := C_{i-1}, \mathbf{a}_i + C_{i-1}, 2\mathbf{a}_i + C_{i-1}, 3\mathbf{a}_i + C_{i-1};$ otherwise $C_i := C_{i-1};$

4. $i := i + 1$; return to 2.

For $0 < i \leq n$, the codes C_i are forced to be linear because we take all linear combinations of the selected vectors $\mathbf{a}_{i1}, \dots, \mathbf{a}_{il}$; $l \leq i$. The codes C_i have a generating set formed by the selected vectors $\mathbf{a}_{i1}, \dots, \mathbf{a}_{il}$.

Considering the greedy algorithm [22, Algorithm A] for finite fields, a natural question that arises is, can a vector $\mathbf{x} \in V_i \setminus V_{i-1}$ exist with $P[\mathbf{x} + \mathbf{c}]$ for all $\mathbf{c} \in C_i$ and $\mathbf{x} \notin C_i$. The following lemma, which is an extension of [22, Theorem 2.1], shows that such a vector \mathbf{x} does not exist.

Lemma 1 *Let P be a multiplicative property over \mathbb{Z}_4 , and let $\mathbf{a}_i \in V_i$ be such that $P[\mathbf{a}_i + \mathbf{c}]$ and $P[2\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1}$, for $i \geq 1$. Then every $\mathbf{x} \in V_i \setminus V_{i-1}$ satisfying $P[\mathbf{x} + \mathbf{c}]$ and $P[2\mathbf{x} + \mathbf{c}]$ for all $\mathbf{c} \in C_i$ is in C_i .*

Proof. The proof is by induction on i . Let $j > 0$ be the first index such that $P[\mathbf{a}_j]$ and $P[2\mathbf{a}_j]$. Hence $C_0 = C_1 = \dots = C_{j-1} = 0$, $C_j = 0, \mathbf{a}_j, 2\mathbf{a}_j, 3\mathbf{a}_j$. Let $\mathbf{x} \in V_j \setminus V_{j-1}$ be a vector such that $P[\mathbf{x} + \alpha \mathbf{a}_j]$ and $P[2\mathbf{x} + \alpha \mathbf{a}_j]$ for $0 \leq \alpha \leq 3$. Since $\mathbf{x} \in V_j \setminus V_{j-1}$, we can write $\mathbf{x} = \beta \mathbf{a}_j + \mathbf{v}$ for some $\beta \neq 0$ and some $\mathbf{v} \in V_{j-1}$. If $\mathbf{v} = 0$, then we have $\mathbf{x} = \beta \mathbf{a}_j$, and hence $\mathbf{x} \in C_j$. If $\mathbf{v} \neq 0$, we first take $\alpha = -\beta$ and obtain $P[\mathbf{v}]$, then take $\alpha = -2\beta$ and obtain $P[2\mathbf{v}]$. This contradicts the assumption on j .

Let $\mathbf{a}_i \in V_i$, $i > j$, be a selected vector such that $P[\mathbf{a}_i + \mathbf{c}]$ and $P[2\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1}$. Assume that the lemma holds for all relevant index values less than i . Now let $\mathbf{x} \in V_i \setminus V_{i-1}$ such that $P[\mathbf{x} + \mathbf{c}]$ and $P[2\mathbf{x} + \mathbf{c}]$ for all $\mathbf{c} \in C_i$. Since $\mathbf{x} \in V_i \setminus V_{i-1}$, we can write $\mathbf{x} = \beta \mathbf{a}_i + \mathbf{v}$ for some $\mathbf{v} \in V_{i-1}$ and

$\beta \neq 0$. If we take $\mathbf{c} = -\beta\mathbf{a}_i + \mathbf{c}'$ and $\mathbf{c} = -2\beta\mathbf{a}_i + \mathbf{c}'$, it follows that $P[\mathbf{v} + \mathbf{c}']$ and $P[2\mathbf{v} + \mathbf{c}']$ for all $\mathbf{c}' \in C_{i-1}$. From the induction assumption we have that $\mathbf{v} \in C_{i-1}$. Since $\mathbf{x} = \beta\mathbf{a}_i + \mathbf{v}$, it must be that $\mathbf{x} \in C_i$. ■

Lemma 1 shows that when a vector $\mathbf{a}_i \in V_i$ is found in Step 2 of Algorithm A, and after extending the list of codewords by Step 3, we can continue the selection procedure by searching the sublist $V_{i+1} \setminus V_i$. Thus at the end of Algorithm A we have a nested sequence of linear codes

$$0 = C_0 \subset C_1 \subset \dots \subset C_n.$$

The set $B = \{\mathbf{a}_{i1}, \dots, \mathbf{a}_{il}\}$ is a generating set for the code C_i . The code C_n is the so-called lexicode and since the codes C_i depend only on the selection property P and the ordering B , the code C_n is denoted by $C(B, P)$. The lexicode $C(B, P)$ is a maximal code in the sense that it cannot be contained in a larger code with the same generating set and the same property.

Remark 1 *Our definition of the multiplicative property differs from that of van Zanten and Nengah Suparta [22]. They defined a multiplicative property over a finite field as a boolean valued function P for which $P[\mathbf{x}]$ implies $P[\alpha\mathbf{x}]$ for all $\alpha \in \mathbb{F}_q$. Since $P[\mathbf{a}_i + \mathbf{c}]$ holds, then $P[\mathbf{a}_i + \alpha\mathbf{c}]$ from Step 2 of [22, Algorithm A]. If the property P is multiplicative, then $P[\alpha^{-1}(\mathbf{a}_i + \alpha\mathbf{c})] = P[\alpha^{-1}\mathbf{a}_i + \mathbf{c}]$ for all $\alpha \in F_q$. This is no longer true over rings since there are zero divisors. Hence there are some vectors $\mathbf{c} \in C_i$ which are missing and may not satisfy the property P even if the code is linear and the property is*

multiplicative. This justifies our modification of the multiplicative property and adding the constraint in Step 2 to also satisfy $P[2\mathbf{a}_i + \mathbf{c}]$.

Now we extend [22, Theorem 2.2] to our lexicode over \mathbb{Z}_4 .

Theorem 2 *For any basis B of \mathbb{Z}_4^n and any multiplicative selection criteria P , the lexicode $C(B, P)$ is linear and $P[\mathbf{x}]$ holds for each codeword $\mathbf{x} \neq 0$.*

Proof. The linearity of the code is assured by the code construction. Since $P[\mathbf{a}_i + \mathbf{c}]$, $P[2\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1}$, and the property P is multiplicative, then for all $\mathbf{c} \in C_{i-1}$, we also have $P[3\mathbf{a}_i + 3\mathbf{c}]$, and $P[2\mathbf{a}_i + 3\mathbf{c}]$. Since C_{i-1} is linear, this is equivalent to having that $P[\alpha\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1}$. Applying this result for $i = 1, 2, \dots, k$ sequentially yields that $P[\mathbf{x}]$ is true for any codeword $\mathbf{x} \neq 0$, since the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ constitute a generating set for the code $C(B, P)$. ■

4 Self-Orthogonal Codes

Let $\mathbf{x} = x_1 \dots x_n$ and $\mathbf{y} = y_1 \dots y_n$ be two elements of \mathbb{Z}_4^n . The inner product of \mathbf{x} and \mathbf{y} in \mathbb{Z}_4^n is defined as $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n \pmod{4}$. Let C be a \mathbb{Z}_4 linear code of length n . The dual code of C is defined as $C^\perp = \{\mathbf{x} \in \mathbb{Z}_4^n \mid \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C\}$. A code is said to be self-orthogonal if $C \subset C^\perp$.

Consider now the property $P[\mathbf{x}]$ is true if and only if $\mathbf{x} \cdot \mathbf{x} = 0$. This is a multiplicative property over \mathbb{Z}_4 because $3\mathbf{x} \cdot 3\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = 0$. Using Algorithm A and this selection property, we produce a linear lexicode $C(B, P)$ over

Table 1: Lexicodes over \mathbb{Z}_4^n with the Selection Property $\mathbf{x} \cdot \mathbf{x} = 0$

n	Basis of \mathbb{Z}_4^n	Basis of $C(B, P)$	Type	d_L
4	Canonical basis	$\mathbf{a}_1 = 2000$ $\mathbf{a}_2 = 0200$ $\mathbf{a}_3 = 0020$ $\mathbf{a}_4 = 1111$	42^3	2
4	$\mathbf{b}_1 = 0001$ $\mathbf{b}_2 = 1100$ $\mathbf{b}_3 = 0110$ $\mathbf{b}_4 = 0011$	$\mathbf{a}_1 = 0002$ $\mathbf{a}_2 = 2200$ $\mathbf{a}_3 = 0220$ $\mathbf{a}_4 = 1111$	42^3	2
6	Canonical basis	$\mathbf{a}_1 = 200000$ $\mathbf{a}_2 = 020000$ $\mathbf{a}_3 = 002000$ $\mathbf{a}_4 = 111100$ $\mathbf{a}_5 = 000020$ $\mathbf{a}_6 = 110011$	$4^2 2^4$	2
6	$\mathbf{b}_1 = 322323$ $\mathbf{b}_2 = 220033$ $\mathbf{b}_3 = 311201$ $\mathbf{b}_4 = 322122$ $\mathbf{b}_5 = 212130$ $\mathbf{b}_6 = 231230$	$\mathbf{a}_1 = 200202$ $\mathbf{a}_2 = 000022$ $\mathbf{a}_3 = 311201$ $\mathbf{a}_4 = 102111$ $\mathbf{a}_5 = 020220$ $\mathbf{a}_6 = 022020$	$4^5 2$	2
8	Canonical basis	$\mathbf{a}_1 = 20000000$ $\mathbf{a}_2 = 02000000$ $\mathbf{a}_3 = 00200000$ $\mathbf{a}_4 = 11110000$ $\mathbf{a}_5 = 00002000$ $\mathbf{a}_6 = 11001100$ $\mathbf{a}_7 = 10000001$ $\mathbf{a}_8 = 01101001$	$4^4 2^4$	2

\mathbb{Z}_4^n . Hence from Theorem 2 we have that the code $C(B, P)$ for this criteria is linear and $P[\mathbf{x}]$ holds for all $\mathbf{x} \in C(B, P)$. In the case of lexicodes over fields, this selection criteria is sufficient to produce self-orthogonal lexicodes. However this is not the case over \mathbb{Z}_4 , since the argument of [22, Corollary 5.1] is not true over rings, namely we can have $\mathbf{x} \cdot \mathbf{x} = 0$, $\mathbf{y} \cdot \mathbf{y} = 0$ and $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = 0$ without having $\mathbf{x} \cdot \mathbf{y} = 0$. However, this criteria may result in a self-orthogonal code. For instance, the first code in Table 1 is self-orthogonal, whereas the second code is not.

Lemma 3 *The property $P[\mathbf{x}]$ is true if and only if $w_E(\mathbf{x}) \equiv 0 \pmod{8}$ is a multiplicative property over \mathbb{Z}_4^n .*

Proof. Let $\mathbf{x} \in \mathbb{Z}_4^n$ such that $w_E(\mathbf{x}) = n_1(\mathbf{x}) + 4n_2(\mathbf{x}) + n_3(\mathbf{x}) \equiv 0 \pmod{8}$. We must prove that $w_E(3\mathbf{x}) \equiv 0 \pmod{8}$. We have $w_E(\mathbf{x}) = w_E(3\mathbf{x})$, because $n_1(3\mathbf{x}) = n_3(\mathbf{x})$, $n_3(3\mathbf{x}) = n_1(\mathbf{x})$ and $n_2(3\mathbf{x}) = n_2(\mathbf{x})$. Since we have assumed that $w_E(\mathbf{x}) \equiv 0 \pmod{8}$, the result follows. ■

Remark 2 *The condition given in Lemma 3 is a sufficient condition to obtain self-orthogonal codes over \mathbb{Z}_4 [17, Theorem 12.2.4]. Hence by applying Algorithm A with the property $P[\mathbf{x}] = w_E(\mathbf{x}) \equiv 0 \pmod{8}$, we obtain self-orthogonal codes. Some of these codes will be self-dual. The minimum Lee distance of the codes are compared with those given in [2] and [12]. The symbol \diamond denotes that there is no result to compare with, and \times denotes that the code is not self-dual.*

5 Lexicodes with a Weight Criteria

As mentioned in the introduction, the first lexicodes were obtained using a weight criteria over finite fields. We now prove that the weight criteria is a multiplicative property.

Lemma 4 *Let δ be a positive integer. The property $P[\mathbf{x}]$ if and only if $w_L(\mathbf{x}) \geq \delta$ is a multiplicative property.*

Table 2: Self-orthogonal Lexicodes over \mathbb{Z}_4^n with the Selection Property
 $w_E(\mathbf{x}) \equiv 0 \pmod{8}$

n	Basis of \mathbb{Z}_4^n	Basis of $C(B, P)$	Type	d_L	[2]	[12]	Self-dual
4	$\mathbf{b}_1 = 0001$ $\mathbf{b}_2 = 1100$ $\mathbf{b}_3 = 0110$ $\mathbf{b}_4 = 0011$	$\mathbf{a}_1 = 2200$ $\mathbf{a}_2 = 0220$ $\mathbf{a}_3 = 0022$	2^3	4	\diamond	4	\times
5	$\mathbf{b}_1 = 01010$ $\mathbf{b}_2 = 10100$ $\mathbf{b}_3 = 33100$ $\mathbf{b}_4 = 00003$ $\mathbf{b}_5 = 00100$	$\mathbf{a}_1 = 02020$ $\mathbf{a}_2 = 20200$ $\mathbf{a}_3 = 22000$ $\mathbf{a}_4 = 11112$	$2^3 4$	4	4	4	\times
6	Canonical basis	$\mathbf{a}_1 = 220000$ $\mathbf{a}_2 = 202000$ $\mathbf{a}_3 = 200200$ $\mathbf{a}_4 = 200020$ $\mathbf{a}_5 = 200002$	2^5	4	\diamond	4	\times
6	$\mathbf{b}_1 = 322323$ $\mathbf{b}_2 = 220033$ $\mathbf{b}_3 = 311201$ $\mathbf{b}_4 = 322122$ $\mathbf{b}_5 = 212130$ $\mathbf{b}_6 = 231230$	$\mathbf{a}_1 = 000022$ $\mathbf{a}_2 = 222002$ $\mathbf{a}_3 = 102111$ $\mathbf{a}_4 = 222222$	$4^1 2^3$	4	\diamond	4	\times
8	$\mathbf{b}_1 = 32121211$ $\mathbf{b}_2 = 01132301$ $\mathbf{b}_3 = 23002111$ $\mathbf{b}_4 = 22231202$ $\mathbf{b}_5 = 11200323$ $\mathbf{b}_6 = 01312220$ $\mathbf{b}_7 = 20121213$ $\mathbf{b}_8 = 31012112$	$\mathbf{a}_1 = 22022220$ $\mathbf{a}_2 = 02000222$ $\mathbf{a}_3 = 00022000$ $\mathbf{a}_4 = 22000202$ $\mathbf{a}_5 = 22022022$ $\mathbf{a}_6 = 00202022$ $\mathbf{a}_7 = 13331313$	$4^1 2^6$	4	\diamond	\diamond	Self-dual
8	$\mathbf{b}_1 = 11112233$ $\mathbf{b}_2 = 23100323$ $\mathbf{b}_3 = 02222133$ $\mathbf{b}_4 = 01133231$ $\mathbf{b}_5 = 21310130$ $\mathbf{b}_6 = 23101130$ $\mathbf{b}_7 = 23001233$ $\mathbf{b}_8 = 11203211$	$\mathbf{a}_1 = 22220022$ $\mathbf{a}_2 = 02200202$ $\mathbf{a}_3 = 02222022$ $\mathbf{a}_4 = 02220002$ $\mathbf{a}_5 = 11131331$ $\mathbf{a}_6 = 02002022$ $\mathbf{a}_7 = 22002200$	$4^1 2^6$	4	\diamond	\diamond	Self-dual
9	$\mathbf{b}_1 = 121221011$ $\mathbf{b}_2 = 232312211$ $\mathbf{b}_3 = 010102101$ $\mathbf{b}_4 = 131023121$ $\mathbf{b}_5 = 233011332$ $\mathbf{b}_6 = 300221122$ $\mathbf{b}_7 = 103131120$ $\mathbf{b}_8 = 222032231$ $\mathbf{b}_9 = 210312111$	$\mathbf{a}_1 = 222222000$ $\mathbf{a}_2 = 010102101$ $\mathbf{a}_3 = 320102312$ $\mathbf{a}_4 = 002000222$ $\mathbf{a}_5 = 000200200$ $\mathbf{a}_6 = 222002022$	$4^2 2^4$	4	\diamond	\diamond	\times

Proof. We need to prove that if $w_L(\mathbf{x}) \geq \delta$ then $w_L(3\mathbf{x}) \geq \delta$. It is easy to see that $n_1(3\mathbf{x}) = n_3(\mathbf{x})$, $n_3(3\mathbf{x}) = n_1(\mathbf{x})$ and $n_2(3\mathbf{x}) = n_2(\mathbf{x})$. This gives that $w_L(3\mathbf{x}) = w_L(\mathbf{x})$. Hence the result follows. ■

For δ a positive integer, from Lemma 4 the property $w_L(\mathbf{x}) \geq \delta$ is multiplicative. Therefore Algorithm A with the property $P[\mathbf{x}]$ if and only if $w_L(\mathbf{x}) \geq \delta$ also produces a nested sequence $0 = C_0 \subset C_1 \subset \dots \subset C_n$ of linear codes. The lexicode is then the code $C_n = C(B, \delta)$. It is clear that C_i is a code with generating set $\{\mathbf{a}_{i1}, \dots, \mathbf{a}_{in}\}$.

Corollary 5 *The lexicode $C(B, \delta)$ given by Algorithm A for designed distance δ is a linear code over \mathbb{Z}_4 with minimum distance greater than or equal to δ .*

Remark 3 *We remark that the selection property on the Lee weight gives codes with good properties. The code of Table 4 in the row case 7 is the self-dual octacode. In the next section we will prove also that their binary image are good.*

6 Good Binary Codes from Lexicodes over \mathbb{Z}_4

It was proved by Hammons et al. [16] that some of the best known nonlinear binary codes such as the Nordstrom-Robinson, Kerdock, Preparata, Goetals and Delsarte-Goetals codes are Gray map images of \mathbb{Z}_4 -linear codes. The

Table 3: Lexicodes over \mathbb{Z}_4^n with the Selection Property $w_L(\mathbf{x}) \geq \delta$

n	Basis of \mathbb{Z}_4^n	δ	Basis of $C(B, P)$	Type	d_L
3	Canonical basis	2	$\mathbf{a}_1 = 110$ $\mathbf{a}_2 = 101$	4^{22}	2
4	$\mathbf{b}_1 = 0001$ $\mathbf{b}_2 = 1100$ $\mathbf{b}_3 = 0110$ $\mathbf{b}_4 = 0011$	2	$\mathbf{a}_1 = 1100$ $\mathbf{a}_2 = 0110$ $\mathbf{a}_3 = 0011$	4^{32}	2
5	Canonical basis	3	$\mathbf{a}_1 = 11100$ $\mathbf{a}_2 = 21010$ $\mathbf{a}_3 = 31001$	4^3	3
5	$\mathbf{b}_1 = 10100$ $\mathbf{b}_2 = 10010$ $\mathbf{b}_3 = 33100$ $\mathbf{b}_4 = 00003$ $\mathbf{b}_5 = 00100$	3	$\mathbf{a}_1 = 11110$ $\mathbf{a}_2 = 33103$	4^2	3
6	Canonical basis	4	$\mathbf{a}_1 = 211000$ $\mathbf{a}_2 = 12011$ $\mathbf{a}_3 = 200011$	4^3	4
6	$\mathbf{b}_1 = 231311$ $\mathbf{b}_2 = 122322$ $\mathbf{b}_3 = 122101$ $\mathbf{b}_4 = 211321$ $\mathbf{b}_5 = 110321$ $\mathbf{b}_6 = 132023$	2	$\mathbf{a}_1 = 231311$ $\mathbf{a}_2 = 122322$ $\mathbf{a}_2 = 122101$ $\mathbf{a}_2 = 312221$	4^4	2
6	”	3	$\mathbf{a}_1 = 231311$ $\mathbf{a}_2 = 122101$ $\mathbf{a}_3 = 333203$	4^3	4
6	”	4	$\mathbf{a}_1 = 231311$ $\mathbf{a}_2 = 122101$ $\mathbf{a}_3 = 210001$	4^3	4
6	”	5	$\mathbf{a}_1 = 231311$ $\mathbf{a}_2 = 122101$	4^2	5
6	”	6	$\mathbf{a}_1 = 231311$	4	7
8	$\mathbf{b}_1 = 22312221$ $\mathbf{b}_2 = 11311303$ $\mathbf{b}_3 = 00121200$ $\mathbf{b}_4 = 01313032$ $\mathbf{b}_5 = 30122132$ $\mathbf{b}_6 = 03213232$ $\mathbf{b}_7 = 32132232$ $\mathbf{b}_8 = 12201321$	5	$\mathbf{a}_1 = 22312221$ $\mathbf{a}_2 = 11311303$ $\mathbf{a}_3 = 01030232$	4^3	5
8	$\mathbf{b}_1 = 11112233$ $\mathbf{b}_2 = 23100323$ $\mathbf{b}_3 = 02222133$ $\mathbf{b}_4 = 01133231$ $\mathbf{b}_5 = 21310130$ $\mathbf{b}_6 = 23101130$ $\mathbf{b}_7 = 23001233$ $\mathbf{b}_8 = 11203211$	2	$\mathbf{a}_1 = 11112233$ $\mathbf{a}_2 = 23100323$ $\mathbf{a}_3 = 02222133$ $\mathbf{a}_4 = 01133231$ $\mathbf{a}_5 = 21310130$ $\mathbf{a}_6 = 20311130$ $\mathbf{a}_7 = 22301233$	4^7	2

Table 4: Lexicodes over \mathbb{Z}_4^n with the Selection Property $w_L(\mathbf{x}) \geq \delta$

n	Basis of \mathbb{Z}_4^n	δ	Basis of $C(B, P)$	Type	d_L
8		3	$\mathbf{a}_1 = 11112233$ $\mathbf{a}_2 = 23100323$ $\mathbf{a}_3 = 02222133$ $\mathbf{a}_4 = 21310130$ $\mathbf{a}_5 = 22133112$	4^5	3
8	"	4	$\mathbf{a}_1 = 11112233$ $\mathbf{a}_2 = 23100323$ $\mathbf{a}_3 = 02222133$ $\mathbf{a}_4 = 23132112$	4^4	4
8		5	$\mathbf{a}_1 = 11112233$ $\mathbf{a}_2 = 23100323$ $\mathbf{a}_3 = 02222133$	4^3	5
8		6	$\mathbf{a}_1 = 11112233$ $\mathbf{a}_2 = 23100323$ $\mathbf{a}_3 = 33033123$	4^3	6
8		7	$\mathbf{a}_1 = 11112233$	4^1	10
8	$\mathbf{b}_1 = 10003121$ $\mathbf{b}_2 = 01001231$ $\mathbf{b}_3 = 00103332$ $\mathbf{b}_4 = 00012311$ $\mathbf{b}_5 = 22233221$ $\mathbf{b}_6 = 10302221$ $\mathbf{b}_7 = 10312111$ $\mathbf{b}_8 = 02311100$	2	$\mathbf{a}_1 = 10003121$ $\mathbf{a}_2 = 01001231$ $\mathbf{a}_3 = 00103332$ $\mathbf{a}_4 = 00012311$ $\mathbf{a}_5 = 22233221$ $\mathbf{a}_6 = 10302221$	4^6	2
8		$3 \leq \delta \leq 6$	$\mathbf{a}_1 = 10003121$ $\mathbf{a}_2 = 01001231$ $\mathbf{a}_3 = 00103332$ $\mathbf{a}_4 = 00012311$	4^4	6
8		7	$\mathbf{a}_1 = 21102321$ $\mathbf{a}_2 = 10310132$	4^2	7
8		8	$\mathbf{a}_1 = 21102321$ $\mathbf{a}_2 = 21213100$	4^2	8
9	$\mathbf{b}_1 = 121221011$ $\mathbf{b}_2 = 232312211$ $\mathbf{b}_3 = 010102101$ $\mathbf{b}_4 = 131023121$ $\mathbf{b}_5 = 233011332$ $\mathbf{b}_6 = 300221122$ $\mathbf{b}_7 = 103131120$ $\mathbf{b}_8 = 222032231$ $\mathbf{b}_9 = 210312111$	8	$\mathbf{a}_1 = 121221011$ $\mathbf{a}_2 = 323311112$	4^2	8
10	$\mathbf{b}_1 = 2212122203$ $\mathbf{b}_2 = 0123002220$ $\mathbf{b}_3 = 0023010100$ $\mathbf{b}_4 = 1010312112$ $\mathbf{b}_5 = 2111023221$ $\mathbf{b}_6 = 1211332321$ $\mathbf{b}_7 = 3110131311$ $\mathbf{b}_8 = 0313130000$ $\mathbf{b}_9 = 1202313120$ $\mathbf{b}_{10} = 1122001000$	8	$\mathbf{a}_1 = 2331120023$ $\mathbf{a}_2 = 0302111120$ $\mathbf{a}_3 = 3001103202$	4^3	8

Gray map from \mathbb{Z}_4 to \mathbb{F}_2^2 is defined as

$$\mathcal{G}'(0) = 00, \mathcal{G}'(1) = 01, \mathcal{G}'(2) = 11, \mathcal{G}'(3) = 10.$$

Then the Gray map $\mathcal{G} : \mathbb{Z}_4^n \longrightarrow \mathbb{F}_2^{2n}$ is defined as

$$\mathcal{G}(a_1, \dots, a_n) = (\mathcal{G}'(a_1), \dots, \mathcal{G}'(a_n)).$$

The following result is well known.

Lemma 6 *The Gray map \mathcal{G} is the distance-preserving map*

$$(\mathbb{Z}_4^n, \text{ Lee distance }) \longrightarrow (\mathbb{F}_2^{2n}, \text{ Hamming distance }).$$

The covering radius of a code C over \mathbb{Z}_4 with respect to the Lee distance is defined as

$$\rho_L(C) = \max_{u \in \mathbb{Z}_4^n} \{ \min_{c \in C} d_L(u, c) \}$$

For $u \in \mathbb{Z}_4^n$, the coset of C is defined to be the set $u + C = \{u + c | c \in C\}$.

A minimum weight vector in a coset is called a coset leader. It is obvious that the covering radius of C with respect to the Lee weight is the largest minimum weight among all cosets.

Lemma 7 ([1, Proposition 3.2]) *Let C be a code over \mathbb{Z}_4 with $\mathcal{G}(C)$ the*

Gray map image of C . Then

$$\rho_L(C) = \rho(\mathcal{G}(C)).$$

Proposition 8 *Let $0 = C_0 \subset C_1 \subset \dots \subset C_n = C(B, \delta)$ be the set of nested codes obtained by Algorithm B for designed distance δ . Hence if the $C_i \subsetneq C_n$ are of type $4^{k_{i_1}}2^{k_{i_2}}$, the covering radius $\rho_L(C_i)$ satisfies*

$$\delta \leq \rho_L(C_i) \leq 2(n - k_{i_1}) - k_{i_2}. \quad (1)$$

Then we have

$$\lfloor \delta/2 \rfloor \leq \lfloor d/2 \rfloor \leq \rho_L(C(B, \delta)) \leq \delta - 1 \leq d - 1. \quad (2)$$

Proof. Assume that $C_i \subsetneq C(B, \delta)$ for some $1 < i < n$. Now, let $x \in C_n \setminus C_i$ be a codeword of minimum weight. Such a vector must be a coset leader of C_i , as $C_i \subsetneq C(B, \delta)$. Hence $\rho_L(C_i) \geq wt_L(x)$ and then $\rho_L(C_i) \geq \delta$. The right side of (1) is obtained from the redundancy bound [1, Theorem 4.6]. Since each vector in \mathbb{Z}_4^n has distance $\delta - 1$ or less to some vector in C_n , the covering radius of C_n is at most $\delta - 1$. By the construction we have $\lfloor \delta/2 \rfloor \leq \lfloor d/2 \rfloor$. The left side of (2) is obtained from the packing radius bound [1, Theorem 4.3]. ■

Theorem 9 *Let $C_L(B, \delta)$ be the lexicode obtained by Algorithm A. Then the binary code $\mathcal{G}(C_L(B, \delta))$ obtained from $C_L(B, \delta)$ by the Gray image meets the*

Gilbert bound.

Proof. Assume that $\mathcal{G}(C_n)$ is a binary code of minimum distance d , which is the same as the minimum distance of $C_L(B, \delta)$ since the Gray map is a weight preserving map. Hence we have $d \geq \delta$, and by Lemma 7 $\rho(\mathcal{G}(C_L(B, \delta))) = \rho_L(C_L(B, \delta))$. Then from Proposition 8 we have $\rho_L(C_L(B, \delta)) \leq \delta - 1$. Since $\delta \leq d$, $\mathcal{G}(C_n)$ has covering radius less than $d_L - 1$. It is well known [17, p. 87], that a code over \mathbb{F}_q with minimum distance d and covering radius $d - 1$ or less meets the Gilbert bound. ■

7 Construction of Lexicodes over $\mathbb{F}_2 + u\mathbb{F}_2$

In this section for simplicity we denote the ring $\mathbb{F}_2 + u\mathbb{F}_2$ by R . There is a Gray-map Φ that is an \mathbb{F}_2 -linear isometry from $(R^n, \text{Lee distance})$ to $(\mathbb{F}_2^{2n}, \text{Hamming distance})$, and is given by $\Phi(x + uy) = (y, x + y)$. An interesting fact about the Gray-map over R is that the image of a self-dual linear code C over R is a self-dual linear binary code [11].

We have that R^n is an R -linear code with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. As with \mathbb{Z}_4 , with respect to this basis we recursively define a lexicographically ordered list $V_i = \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{4^i}$ as follows

$$\begin{aligned} V_0 &:= 0 \\ V_i &:= V_{i-1} \quad , \mathbf{b}_i + V_{i-1}, u\mathbf{b}_i + V_{i-1}, \overline{u}\mathbf{b}_i + V_{i-1}, 1 \leq i \leq n. \end{aligned}$$

In this way we can identify R^n by V_n . Assume now that we have a property P which can test if a vector $\mathbf{c} \in R^n$ is selected or not. This property is said to be multiplicative if for $P[\mathbf{x}]$ then $P[\overline{u}\mathbf{x}]$. Furthermore, assume that this property is multiplicative. The following greedy algorithm provides lexicode over R^n .

Algorithm B

1. $C_0 := 0; i := 1;$
2. select the first vector $\mathbf{a}_i \in V_i \setminus V_{i-1}$ such that $P[\mathbf{a}_i + \mathbf{c}]$ and $P[u\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_{i-1};$
3. if such an \mathbf{a}_i exists, then $C_i := C_{i-1}, \mathbf{a}_i + C_{i-1}, u\mathbf{a}_i + C_{i-1}, \overline{u}\mathbf{a}_i + C_{i-1};$ otherwise $C_i := C_{i-1};$
4. $i := i + 1;$ return to 2.

There is a similarity between Algorithms A and B so that we can identify 0 by 0, 1 by 1, 2 by u and 3 by \overline{u} . This provides the following result.

Lemma 10 (cf. Lemma 1) *Let $\mathbf{a}_i \in V_i$ be such that $P[\mathbf{a}_i + \mathbf{c}]$ and $P[u\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_i$, for $i \geq 1$. Then every $\mathbf{x} \in V_i \setminus V_{i-1}$ satisfying $P[\mathbf{x} + \mathbf{c}]$ and $P[u\mathbf{a}_i + \mathbf{c}]$ for all $\mathbf{c} \in C_i$ is in C_i .*

At the end of Algorithm B we have a nested sequence of linear codes

$$0 = C_0 \subset C_1 \subset \cdots \subset C_n.$$

The set $B = \{\mathbf{a}_{i1}, \dots, \mathbf{a}_{il}\}$ is a generating set for the code C_i . The code C_n is the so-called lexicode.

Theorem 11 (cf. Theorem 2) *For any basis B of R^n and any multiplicative selection criterion P , the lexicode $C(B, P)$ is linear and $P[\mathbf{x}]$ holds for each codeword $\mathbf{x} \neq 0$.*

We now prove the following result.

Proposition 12 *Let C be a self-orthogonal code over R . Then for all $\mathbf{x} \in C$ we have $w_L(\mathbf{x}) \equiv 0 \pmod{2}$.*

Proof. Let C be a self-orthogonal code and \mathbf{x} be a codeword of C . Then we have $\mathbf{x} \cdot \mathbf{x} = 0 \pmod{2}$, but

$$\mathbf{x} \cdot \mathbf{x} = \sum_1^n x_i x_i = \sum_1^{n_1(\mathbf{x})} 1 + \sum_1^{n_u(\mathbf{x})} u^2 + \sum_1^{n_{\bar{u}}(\mathbf{x})} \bar{u}^2 = n_1(\mathbf{x}) + n_{\bar{u}}(\mathbf{x}) \equiv 0 \pmod{2}.$$

This gives $w_L(\mathbf{x}) \equiv 0 \pmod{2}$. ■

Lemma 13 *The property $P[\mathbf{x}]$ is true if and only if $w_L(\mathbf{x}) \equiv 0 \pmod{2}$ is a multiplicative property.*

Proof. Let \mathbf{x} in R^n be such that $w_L(\mathbf{x}) \equiv 0 \pmod{2}$. We have $n_1(\bar{u}\mathbf{x}) = n_{\bar{u}}(\mathbf{x})$, $n_u(\bar{u}\mathbf{x}) = n_u(\mathbf{x})$, and $n_{\bar{u}}(\bar{u}\mathbf{x}) = n_1(\mathbf{x})$. Hence $w_L(\bar{u}\mathbf{x}) = w_L(\mathbf{x}) \equiv 0 \pmod{2}$. ■

Table 5: Lexicodes over R^n with the Selection Property $w_L(\mathbf{x}) \equiv 0 \pmod{2}$

n	Basis of R^n	Basis of $C(B, P)$	Type	d_L
4	Canonical basis	$a_1 = u000$ $a_2 = 1010$ $a_3 = 1100$ $a_4 = 1001$	$4^3 2$	2
4	$b_1 = 1100$ $b_2 = 1u01$ $b_3 = \overline{u}11\overline{u}$ $b_4 = \overline{u}\overline{u}0\overline{u}$	$a_1 = 1100$ $a_2 = 1u01$ $a_3 = \overline{u}11\overline{u}$ $a_4 = uu0u$	$4^3 2$	2
6	$b_1 = 1u\overline{u}0\overline{u}u$ $b_2 = \overline{u}u\overline{u}uu1$ $b_3 = 0\overline{u}\overline{u}u11$ $b_4 = u01u1u$ $b_5 = 1uu101$ $b_6 = 1\overline{u}01u\overline{u}$	$a_1 = u0u0u0$ $a_2 = u00u1\overline{u}$ $a_3 = 0\overline{u}\overline{u}u11$ $a_4 = u01u1u$ $a_5 = 0011\overline{u}\overline{u}$ $a_6 = 1\overline{u}01u\overline{u}$	$4^5 2$	2
6	$b_1 = \overline{u}00u11$ $b_2 = uu0\overline{u}u\overline{u}$ $b_3 = 101110$ $b_4 = 110001$ $b_5 = u01011$ $b_6 = u10uuu$	$a_1 = u000uu$ $a_2 = uu0\overline{u}u\overline{u}$ $a_3 = 101110$ $a_4 = u10u10$ $a_5 = u10u10$ $a_6 = 1100\overline{u}\overline{u}$	$4^5 2$	2
6	$b_1 = 0u0\overline{u}0u$ $b_2 = 0\overline{u}0\overline{u}10$ $b_3 = 10\overline{u}100$ $b_4 = 1\overline{u}110u$ $b_5 = 001100$ $b_6 = 00\overline{u}\overline{u}11$	$a_1 = 000u00$ $a_2 = 01001u$ $a_3 = 1u\overline{u}u0u$ $a_4 = 1\overline{u}110u$ $a_5 = 001100$ $a_6 = 00\overline{u}\overline{u}11$	$4^5 2$	2
8	$b_1 = \overline{u}1111uuu$ $b_2 = 0\overline{u}00u01\overline{u}$ $b_3 = 000uuu11$ $b_4 = u\overline{u}u1\overline{u}111$ $b_5 = u\overline{u}u1\overline{u}110$ $b_6 = 100\overline{u}1\overline{u}u1$ $b_7 = uu\overline{u}\overline{u}11\overline{u}1$ $b_8 = 0\overline{u}u1u1uu$	$a_1 = uuuuu000$ $a_2 = \overline{u}u11\overline{u}u\overline{u}1$ $a_3 = \overline{u}11uu1\overline{u}\overline{u}$ $a_4 = u\overline{u}u1\overline{u}111$ $a_5 = 1u\overline{u}0u\overline{u}\overline{u}u$ $a_6 = u11u010\overline{u}$ $a_7 = uu\overline{u}\overline{u}11\overline{u}1$ $a_8 = \overline{u}u\overline{u}0\overline{u}\overline{u}00$	$4^7 2$	2

7.1 Lexicodes with a Weight Criterion

Algorithm B with a weight criterion also produces a nested sequence $0 = C_0 \subset C_1 \subset \dots \subset C_n$ of linear codes. The lexicode is then the code $C_n = C(B, \delta)$. It is a code with generating set $\{\mathbf{a}_{i1}, \dots, \mathbf{a}_{il}\}$. From Lemma 1 and Theorem 2 we have the following.

Corollary 14 *The lexicode $C(B, \delta)$ given by Algorithm B for designed distance δ is a linear code over R with minimum distance greater than or equal to δ .*

Remark 4 *With few exceptions, the binary images of the codes given in Table 7 are optimal codes according to [15].*

Conclusions

In this paper, greedy algorithms were given to construct linear lexicodes over the rings \mathbb{Z}_4 and $R = \mathbb{F}_2 + u\mathbb{F}_2$. These algorithms can be used to construct optimal and self-dual codes over these rings, and optimal binary codes. Since the image of a self-dual code over \mathbb{Z}_4 is formally self-dual [10], and the image of a self-dual code over R is self-dual, we obtain binary formally self-dual and binary self-dual codes, respectively. Note that the algorithms presented in this paper can easily be generalized to finite chain rings. Another interesting problem is to generalize the algorithms to finite principal ideal rings and more generally to other rings.

Table 6: Lexicodes over R^n with the Selection Property $w_L(\mathbf{x}) \geq \delta$

n	Basis of R^n	δ	Basis of $C(B, P)$	Type	d_L
6	Canonical basis	4	$a_1 = u11000$ $a_2 = 1u0100$ $a_3 = u00011$	4^3	4
6	$b_1 = 1u\bar{u}0\bar{u}u$ $b_2 = \bar{u}u\bar{u}uu1$ $b_3 = 0\bar{u}u11$ $b_4 = u01u1u$ $b_5 = 1uu101$ $b_6 = 1\bar{u}01u\bar{u}$	4	$a_1 = 1u\bar{u}0\bar{u}u$ $a_2 = \bar{u}u\bar{u}uu1$	4^2	4
6	$b_1 = 0u0\bar{u}0u$ $b_2 = 0\bar{u}0\bar{u}10$ $b_3 = 10\bar{u}100$ $b_4 = 1\bar{u}110u$ $b_5 = 001\bar{u}00$ $b_6 = 00\bar{u}u11$	5	$a_1 = 11\bar{u}u1u$ $a_2 = 0\bar{u}u\bar{u}uu1$	4^2	5
6	$b_1 = 0u0\bar{u}0u$ $b_2 = 0\bar{u}0\bar{u}10$ $b_3 = 10\bar{u}100$ $b_4 = 1\bar{u}110u$ $b_5 = 001\bar{u}00$ $b_6 = 00\bar{u}u11$	4	$a_1 = 010u1u$ $a_2 = 1u\bar{u}00u$ $a_3 = 1\bar{u}110u$	4^3	4
6	$b_1 = 0000u1$ $b_2 = \bar{u}00\bar{u}11$ $b_3 = \bar{u}u1100$ $b_4 = u\bar{u}u1\bar{u}0$ $b_5 = \bar{u}1u\bar{u}u1$ $b_6 = u\bar{u}u10u$	5	$a_1 = \bar{u}00\bar{u}u$ $a_2 = \bar{u}u11u1$	4^2	5
6	$b_1 = 0000u1$ $b_2 = \bar{u}00\bar{u}11$ $b_3 = \bar{u}u1100$ $b_4 = u\bar{u}u1\bar{u}0$ $b_5 = \bar{u}1u\bar{u}u1$ $b_6 = u\bar{u}u10u$	4	$a_1 = \bar{u}00\bar{u}11$ $a_2 = \bar{u}u1100$ $a_3 = \bar{u}1u\bar{u}u1$	4^3	4
8	$b_1 = \bar{u}u1u1u11$ $b_2 = 011\bar{u}u\bar{u}01$ $b_3 = u\bar{u}00u111$ $b_4 = uuu\bar{u}1u0u$ $b_5 = 11u00\bar{u}u\bar{u}$ $b_6 = 01\bar{u}1uuu0$ $b_7 = u01u1u1\bar{u}$ $b_8 = \bar{u}101u11u$	5	$a_1 = \bar{u}u1u1u11$ $a_2 = 011\bar{u}u\bar{u}01$ $a_3 = 111u\bar{u}uuu$ $a_4 = \bar{u}\bar{u}1uuu\bar{u}u$	4^4	5

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